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Contact Transformations and Conformal Group. III. Finite Nonrelativistic Transformations[1]

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Abstract

Contact conformal transformations (as defined in a previous work) are integrated giving the finite form of these transformations. Some properties of such a group of motions are pointed out.

We must refer to some already published papers (Boya and Cerveró, 1975a, 1975b). In (1975b) the contact conformal transformations in the nonrelativistic case were defined, in their infinitesimal form. In order to analyze more deeply the apparent unphysical transformations Y_1, Y_2, and Y_3 (1975b), which are pure contact transformations, we use here the usual way of integration for such infinitesimal generators. Finite transformations in the form

$$x \to \bar{x}' = f(\bar{x})$$

can be put as follows:

$$x_i' = x_i + \frac{1}{1!}(Xx_i)\tau + \frac{1}{2!}(X^2x_i)\tau^2 + \cdots + \frac{1}{n!}(X^n x_i)\tau^n \tag{1}$$

where X is the infinitesimal generator of the transformation. In our case

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$x \equiv (x, t, v)$ and $X \equiv (Y_1, Y_2, Y_3)$. Starting with Y_1 we find

$$\begin{cases} Y_1 t = tF \\ Y_1{}^2 t = 2tF^2 \\ \quad \cdot \quad \cdot \quad \cdot \\ \quad \cdot \quad \cdot \quad \cdot \\ Y_1{}^n t = n!tF^n \end{cases} \tag{2}$$

where $F = \frac{1}{2}vt - x$. Therefore, using (1),

$$t' = t + a_1 tF + \cdots + a_1{}^n tF^n + \cdots = \frac{t}{1 - a_1 F} \tag{3}$$

because the right-hand side of (3) is the asymptotic expansion of such a function, since

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n$$

and

$$\frac{1}{(1-x)^2} = -[1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots] \tag{4}$$

With the same procedure for x we find

$$\begin{cases} Y_1 x = F^2 + 2xF \\ Y^2 x = 4F^3 + 6xF^2 \\ \quad \cdot \quad \cdot \quad \cdot \\ \quad \cdot \quad \cdot \quad \cdot \\ Y_1{}^n x = n \cdot n!F^{n+1} + (n+1)!xF^n \end{cases} \tag{5}$$

and applying (1) and making use of (4) we get

$$x' = x\left[1 + \frac{1}{1!}2!a_1F + \frac{1}{2!}3!(a_1F)^2 + \cdots \frac{(a_1F)^n}{n}(n+1)!\right]$$

$$+ F[Fa_1 + 2(Fa_1)^2 + \cdots + n(Fa_1)^n]$$

$$= -\frac{a_1F^2 + x}{(1 - a_1F)^2} \tag{6}$$

Finally for the transformation in the v-coordinate

$$\begin{cases} Y_1 v = vF \\ Y_1^2 v = 2vF^2 \\ \quad \cdot \quad \cdot \quad \cdot \\ \quad \cdot \quad \cdot \quad \cdot \\ Y_1^n v = n! vF^n \end{cases} \tag{7}$$

and we obtain $v' = v/(1 - a_1 F)$ after a short calculation.

Similar procedures must be employed in the integration of the Y_2 and Y_3 infinitesimal generators. For brevity, we give only the answers:

$$\begin{cases} Y_2 t = vt - x \\ Y_2^2 t = v(vt - x) \\ \quad \cdot \quad \cdot \quad \cdot \\ \quad \cdot \quad \cdot \quad \cdot \\ Y_2^n t = \dfrac{n!}{2^{n-1}} v^{n-1}(vt - x) \end{cases} \tag{8}$$

and

$$\begin{cases} Y_2 x = \frac{1}{2} v^2 t \\ Y_2^2 x = \frac{1}{2} v^2 (2vt - x) \\ \quad \cdot \quad \cdot \quad \cdot \\ \quad \cdot \quad \cdot \quad \cdot \\ Y_2^n x = \dfrac{n!}{2^n} v^n [nvt - (n-1)x] \end{cases} \tag{9}$$

and

$$\begin{cases} Y_2 v = \frac{1}{2} v^2 \\ Y_2^2 v = \dfrac{2!}{2^2} v^3 \\ \quad \cdot \quad \cdot \quad \cdot \\ \quad \cdot \quad \cdot \quad \cdot \\ Y_2^n v = \dfrac{n!}{2^n} v^{n+1} \end{cases} \tag{10}$$

and the relevant actuation of Y_3 is only

$$\begin{cases} Y_3 t = v \\ Y_3{}^2 t = 0 \end{cases}$$

$$\begin{cases} Y_3 x = \frac{1}{2}v^2 \\ Y_3{}^2 x = 0 \end{cases} \tag{11}$$

$$\{Y_3 v = 0.$$

The final form of transformations using (8)-(11) and adding the Y_1 transformation is

$$Y_1 \begin{cases} x' = -\dfrac{a_1F^2 + x}{(1 - a_1F)^2} \\ t' = \dfrac{t}{1 - a_1F} \\ v' = \dfrac{v}{1 - a_1F} \end{cases} \tag{12}$$

where $F = \frac{1}{2}vt - x$

$$Y_2 \begin{cases} x' = \dfrac{a_2v^2(t - a_2x)}{(1 - va_2)} + x(1 - va_2) \\ t' = t + \dfrac{2a_2(vt - x)}{(1 - va_2)} \\ v' = \dfrac{v}{(1 - va_2)} \end{cases} \tag{13}$$

and

$$Y_3 \begin{cases} x' = x + \frac{1}{2}a_3v^2 \\ t' = t + va_3 \\ v' = v \end{cases} \tag{14}$$

Let us concentrate on the obvious singular behavior of transformations defined in (12) and (13), since (14) is well defined in the whole space. In the first case this singular behavior occurs for $(1 - a_1F) = 0$ or, in other words,

$$\tfrac{1}{2}vt - x = k_1 \qquad (k_1 = 1/a_1) \tag{15}$$

For each value of k_1 (or $1/a_1$) we have a hyperboloid as the singular locus of the transformations. For the condition (13), (15) takes the simple form

$$v = k_2 \qquad (k_2 = 1/a_2) \tag{16}$$

which is a family of flat surfaces, parallel to the xt plane.

In addition, the asymptotic expansion (4) requires $|x| < 1$. Therefore, in the first case the transformation is only defined if for each value, and consequently for each surface (15), x and t take values according to this expression. A similar condition must be imposed in the second case leading to v values bounded by $|v| < k_2$.

References

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